A note on the regularity property of semi-Markov processes with Borel state space*

Oscar Vega-Amaya†

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Abstract

This note shows that a semi-Markov process with Borel state space is regular under a fairly weak condition on the mean sojourn or holding times and assuming that the embedded Markov chain satisfies either one of the following conditions: (a) it is Harris recurrent; (b) it is recurrent and the “recurrent part” of the state space is reached with probability one for every initial state; (c) it has a unique invariant probability measure. Under the latter condition, the regularity property is only ensured for almost all initial state with respect to the invariant probability measure.

Key words: semi-Markov process, regularity property, recurrence, Harris recurrence, invariant probability measures.

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1 Introduction

A semi-Markov process (SMP) combines the probabilistic structures of a Markov chain and a renewal process; in other words, the SMP makes transitions according to a Markov chain but the time spent between successive

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†Departamento de Matemáticas, Universidad de Sonora, Luis Encinas y Rosales s/n, C. P. 83000, Hermosillo, Sonora, MÉXICO. E-mail: ovega@gauss.mat.uson.mx
transitions are random variables whose distribution functions depend on the “present” state of the system. Thus, it is raised the question of whether the SMP experiences finite or infinitely many transitions in bounded time periods. If the former property holds, the semi-Markov process is said to be regular (or nonexplosive), and irregular (or explosive) otherwise.

A natural way to obtain the regularity property is to impose conditions that guarantee that transitions do not take place too quickly, and the most popular condition to do this is that introduced by Ross [5, Prop. 5.1, p. 88]—see also Çinlar [2, Prop. 3.19, p. 327]. Roughly speaking this condition requires that transition times to be greater than some $\gamma > 0$ with a probability of at least $\epsilon > 0$, independently of the present state of the system. Under this condition, Ross proves the regularity property of the SMP for the countable state space case only, and Bhatthacharya and Majumdar [1] extend the result to Borel spaces. It is worth mentioning that Çinlar’s proof [2, Prop. 3.19, p. 327] extends directly to the general case of Borel spaces.

Moreover, for the countable space case, Ross [5, Prop. 5.1] and Çinlar [2, Cor. 3.17] prove that the regularity property holds whenever the “embedded” Markov chain reaches a recurrent state with probability one for every initial state. Thus, in particular, the regularity property holds if the embedded Markov chain is recurrent. However, their proofs cannot be extended, or at least not directly, to the case of Borel state space because they rely on the renewal process formed by the successive times at which a recurrent state is visited, which typically involves events of probability zero if the state space is uncountable. In fact, to the best of our knowledge, there is no counterpart of these result for Borel spaces.

The aim of the present note is to fill this gap by extending the above results to SMP with Borel state space. More precisely, imposing a fairly weak condition on the sojourn or holding times, we show that the regularity property holds under each one of the following conditions: (a) the embedded Markov chain is Harris recurrent; (b) the embedded Markov chain is recurrent and the “recurrent part” of the state space is reached with probability one for each initial state; (c) the embedded Markov chain has a unique invariant probability measure. Under the latter condition, the regularity property is only ensured for almost all initial state with respect to the invariant probability measure.
2 Preliminary concepts

Let \((X, B)\) be a measurable space where \(X\) is a Borel space and \(B\) is its Borel \(\sigma\)-algebra. We denote by \(\mathbb{R}_+\) and \(\mathbb{N}_0\) the sets of nonnegative real numbers and nonnegative integer, respectively. Let \(Q(\cdot, \cdot)\) be a stochastic kernel on \(X \times \mathbb{R}_+\) given \(X\), and let \(\{(X_n, \delta_{n+1}) : n \in \mathbb{N}_0\}\) be the corresponding Markov chain defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), that is,

\[
Q(B, [0, t]|x) = \mathbb{P}[X_{n+1} \in B, \delta_{n+1} \leq t|X_n = x]
\]

for all \(B \in \mathcal{B}, t \in \mathbb{R}_+, x \in X\).

The process \(\{(X_n, \delta_{n+1}) : n \in \mathbb{N}_0\}\) is called a Markov renewal process and it is usually thought of as a model of a stochastic system evolving as follows: it is observed at time \(t = 0\) in some initial state \(X_0 = x \in X\), in which it remains up to the (nonnegative) random time \(\delta_1\). The distribution function of \(\delta_1\) is given by

\[
F(t|x) := Q(X, [0, t]|x) \quad \forall t \in \mathbb{R}_+, x \in X,
\]

which is called the sojourn or holding time distribution in the state \(x\). Next, at time \(\delta_1\), the system jumps to a new state, say \(X_1 = y \in X\), according to the probability measure

\[
P(B|x) := Q(B, \mathbb{R}_+|x) \quad \forall B \in \mathcal{B}, x \in X.
\]

Once the transition occurs, the system remains in the new state \(X_1 = y\) up to the (nonnegative) random time \(\delta_2\), and so on.

The state of the system is tracked in continuous time by the process

\[
Z_t := X_n \quad \text{if} \quad T_n \leq t < T_{n+1}
\]

where

\[
T_{n+1} := T_n + \delta_{n+1} \quad \forall n \in \mathbb{N}_0, \quad \text{and} \quad T_0 := 0.
\]

The continuous-time process \(\{Z_t : t \in \mathbb{R}_+\}\) is called a semi-Markov process (SMP) with (semi-Markov) kernel \(Q(\cdot, \cdot)\).

Note, by (1), that the process \(\{X_n : n \in \mathbb{N}_0\}\) is a Markov chain on \(X\) with one-step probability transition \(P(\cdot|\cdot)\). Thus, it is called the embedded Markov chain in the semi-Markov process \(\{Z_t : t \in \mathbb{R}_+\}\).
Now observe that the kernel $Q(\cdot, \cdot | \cdot)$ can be “disintegrated” as follows:

$$Q(B, [0,t]|x) = \int_B G(t|x,y)P(dy|x) \quad \forall B \in \mathcal{B}, t \in \mathbb{R}_+, x \in X,$$

where $G(\cdot|x,y)$ is a distribution function on $\mathbb{R}_+$ for all $x,y \in X$, and $G(t|\cdot, \cdot)$ is measurable function on $X \times X$ for each $t \in \mathbb{R}_+$. Thus,

$$G(t|x,y) = \mathbb{P}[\delta_{n+1} \leq t|X_n = x, X_{n+1} = y] \quad \forall x,y \in X, t \in \mathbb{R}_+. \quad (7)$$

Then, using the Markov property of the Markov renewal process and (7), it is easy to prove that the random variables $\{\delta_n : n \in \mathbb{N}\}$ are (conditionally) independent given the state process $\{X_n : n \in \mathbb{N}_0\}$, and also that

$$\mathbb{P} [\delta_1 \leq t_1, \ldots, \delta_n \leq t_n|X_0, X_1, \ldots, X_n] = \prod_{k=1}^n G(t_k|X_{k-1}, X_k). \quad (8)$$

3 The regularity property, recurrence and invariant measures

In the following we shall write $\mathbb{P}_x[\cdot]$ instead of $\mathbb{P}[\cdot|X_0 = x]$, and $\mathbb{E}_x[\cdot]$ stands for the expectation operator with respect to $\mathbb{P}_x[\cdot]$.

A state $x \in X$ is said to be regular if

$$\lim_{n \to \infty} T_n = \infty \quad \mathbb{P}_x-\text{a.s.}$$

The SMP is said to be regular if every state $x \in X$ is regular.

**Assumption 1.** The embedded Markov chain $\{X_n : n \in \mathbb{N}_0\}$ is Harris recurrent, which means that

$$\sum_{n=0}^{\infty} I_A(X_n) = \infty \quad \mathbb{P}_x-\text{a.s.}$$

for all $x \in X$ and all $A \in \mathcal{B}^+ := \{B \in \mathcal{B} : \psi(B) > 0\}$, where $\psi(\cdot)$ is a maximal irreducibility measure.
To state our second condition, let

$$\Delta(x) := \int_{\mathbb{R}_+} \exp(-t) F(dt|x) \quad \forall x \in X.$$ 

Note that $0 < \Delta(\cdot) \leq 1$, and also that $\Delta(x) = 1$ if only if $F(0|x) = 1$. Thus, it is required a condition to exclude this degenerate case to occur for all or “almost all” initial states.

**Assumption 2.** There exists $\alpha \in (0, 1)$ for which the set $B := \{ x \in X : \Delta(x) \leq \alpha \}$ is in $\mathcal{B}^+$. 

**Remark 3.** (a) Note that Assumption 2 holds if $\Delta(x) < 1$ for all $x \in X$. To see that this is true, let $\alpha_n \uparrow 1$ and put $B_n := \{ x \in X : \Delta(x) \leq \alpha_n \}$. Then, $X = \cup_{n=1}^{\infty} B_n$; therefore, there exists $n \in \mathbb{N}_0$ such that $B_n \in \mathcal{B}^+$. 

(b) It follows from the conditional independence of the random variables $\delta_1, \delta_2, \cdots$ and (8) that

$$\mathbb{E}_x [\exp(-T_n+1)|X_0, X_1, \cdots, X_n] = \Delta(X_0) \cdots \Delta(X_n) \quad \forall n \in \mathbb{N}_0.$$

Hence,

$$\lim_{n \to \infty} T_n = \infty \Leftrightarrow [\Delta(X_0) \cdots \Delta(X_n)] \to 0. \quad (9)$$

We now state the first result of this note.

**Theorem 4.** If Assumptions 1 and 2 hold, then the semi-Markov process is regular, that is,

$$\lim_{n \to \infty} T_n = \infty \quad \mathbb{P}_x \text{-a.s.} \forall x \in X. \quad (10)$$

**Proof of Theorem 4.** Let $B$ as in Assumption 1. For each $n = 1, 2, \cdots$, let

$$\sigma(1) := \inf\{ k > 0 : X_k \in B \}, \quad \sigma(n + 1) := \inf\{ k > \sigma_n : X_k \in B \},$$

and

$$S_{n+1} := \sum_{k=0}^{n} I_B(X_k).$$
Now observe that
\[ \Delta(X_0) \cdots \Delta(X_n) \leq \Delta(X_{\sigma(1)}) \Delta(X_{\sigma(2)}) \cdots \Delta(X_{\sigma(S_{n+1})}) \leq \alpha^{S_{n+1}} \]
on the set \([S_{n+1} \neq 0]\). Thus, since \(\psi(B) > 0\), Assumption 1 implies that \(S_{n+1} \to \infty \) \(\mathbb{P}_x\)-a.s. for all \(x \in \mathbf{X}\); hence
\[ \Delta(X_0) \cdots \Delta(X_n) \to 0 \quad \mathbb{P}_x\text{-a.s. } \forall x \in \mathbf{X}, \]
which, by (9), proves that the process is regular. \(\blacksquare\)

The regularity property of the SMP can also be obtained assuming that embedded Markov chains \(\{X_n : n \in \mathbb{N}_0\}\) is \textit{recurrent}, which means that
\[ E_x \sum_{k=0}^{\infty} I_A(X_n) = \infty \quad \forall x \in \mathbf{X}, A \in \mathcal{B}^+. \]
However, as in [5, Prop. 5.1] and [2, Cor. 3.17], it is necessary to assume additionally that the “recurrent part” of the state space is reached with probability one for every initial state. To state this condition precisely, we need the following important result (see [4, Thm 9.0.1] or [3, Prop. 4.2.12]).

\textbf{Remark 5.} If the embedded Markov chains \(\{X_n : n \in \mathbb{N}_0\}\) is recurrent, then
\[ X = H \cup N \]
where the measurable set \(H\) is \textit{full} and \textit{absorbing} (that is, \(\psi(N) = 0\), and \(Q(H|x) = 1\) for all \(x \in \mathbf{X}\), respectively). Moreover, the Markov chain restricted to \(H\) is Harris recurrent, that is,
\[ \sum_{k=0}^{\infty} I_A(X_n) = \infty \quad \mathbb{P}_x\text{-a.s. } \forall x \in H, A \subseteq H, \]
whenever \(A \in \mathcal{B}^+\).

\textbf{Theorem 6.} Suppose Assumption 2 holds. If the embedded Markov chain is recurrent and
\[ \tau := \inf \{n \geq 0 : X_n \in H\} < \infty \quad \mathbb{P}_x\text{-a.s. } \forall x \in \mathbf{X}, \]
then the SMP is regular.
Proof of Theorem 6. The proof follows the same arguments as those given for the proof of Theorem 4 but considering the set \( \overline{B} := B \cap H \) instead of the set \( B \) in Assumption 2.

Note that Theorems 4 and 6 states that the regularity property holds for all initial state \( x \in X \) under a recurrence condition independently of whether the embedded Markov chain admits an invariant probability measure \( \mu(\cdot) \), that is, a probability measure \( \mu(\cdot) \) on \( X \) satisfying the condition

\[
\mu(B) = \int_X P(B|x) \mu(dy) \quad \forall B \in \mathcal{B}.
\]

On the other hand, recurrence (and then Harris recurrence) may be dispensed if one supposes the existence of a unique invariant probability measure with the cost that the regularity property will be ensured only for almost all initial states (see Theorem 9 below). The proof uses a pathwise ergodic theorem which is borrowed from [3, Cor. 2.5.2]. To state this result we need the following notation: for a measurable function \( v \) and measure \( \lambda \) on \( X \), let

\[
\lambda(v) := \int_X v(y) \lambda(dy)
\]

whenever the integral is well defined. Moreover, denote by \( L_1(\lambda) \) the class of measurable functions \( v \) on \( X \) such that \( |v| \) is integrable.

Remark 7.(a) Suppose that \( \{X_n : n \in \mathbb{N}_0\} \) has a unique invariant probability measure \( \mu(\cdot) \). Then, for each function \( v \in L_1(\mu) \) there exists a set \( B_v \in \mathcal{B} \), with \( \mu(B_v) = 1 \), such that

\[
\frac{1}{n} \sum_{k=0}^{n-1} v(X_n) \to \mu(v) \quad \mathbb{P}_x \text{-a.s.} \quad \forall x \in B_v.
\] (11)

(b) If in addition the Markov chain is Harris recurrent, then (11) holds for all \( x \in X \) (see [3, Thm. 4.2.13]).

Assumption 8.(a) The embedded Markov chain has a unique invariant probability measure \( \mu(\cdot) \);

(b) \( \mu(\Delta) = \int_X \Delta(x) \mu(dx) < 1 \).

Theorem 9. If Assumption 8 holds, then the process is regular for \( \mu \)-almost all \( x \in X \). If in addition the embedded Markov chain is Harris recurrent, then the regularity property holds for all \( x \in X \).
Proof of Theorem 9. Observe that
\[
[\Delta(X_0) \cdots \Delta(X_n)]^{1/(n+1)} \leq \frac{1}{n+1} \sum_{k=0}^{n} \Delta(X_k) \quad \forall n \in \mathbb{N}_0.
\]
Thus, by Remark 7(a) and Assumption 8, there exists a set \( B_\Delta \in \mathcal{B} \) such that
\[
\frac{1}{n+1} \sum_{k=0}^{n} \Delta(X_k) \rightarrow \mu(\Delta) < 1 \quad \mathbb{P}_x\text{-a.s.} \quad \forall x \in B_\Delta,
\]
and \( \mu(B_\Delta) = 1 \). Therefore
\[
[\Delta(X_0) \cdots \Delta(X_n)] \rightarrow 0 \quad \mathbb{P}_x\text{-a.s.} \quad \text{for } \mu\text{-almost all } x \in \mathbf{X}.
\]
The second statement of the theorem follows from Theorem 1 because \( \mu(\Delta) < 1 \) implies Assumption 2. \( \blacksquare \)

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References


